VARIANCE SWAPS PRICING

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Abstract

This article describes potential practical approaches to volatility swap pricing. We analyze 3 possible pricing methods based on historic data. First method used historic volatilities, second method uses implied volatilities and third method uses direct option prices.

The best way of instrument's description is example getting. We got example from [1]

Example of a Variance Swap
Bank of America Securities LLC
Indicative Terms
(Discussion Only)

S&P 500 Index Realized Variance Swap

Equity Payer: Bank of America, N.A. ("BofA")

Equity Receiver: Merrill Lynch International

Trade Date: October 8, 1999

Maturity Date: May 7, 2003

Underlying Index: The Standard & Poor's 500 Composite Stock Price Index

Equity Calculations:

(a) "Initial Price" means 0.305

(b) " Final Price" means the actual realized index Variance defined in accordance with the following formula and definition:

$$\sqrt{\frac{\sum_{i=1}^{n-1}\left[\ln\left(\frac{P_{i+1}}{P_i}\right)\right]^2}{n-2}}\times\sqrt{52}$$

(c) "Natural Logarithm" means for any Daily Quotient, as determined by the Calculation Agent, the exponential number which equates 2.718281828 to such Daily Quotient;

(d) "N" means the total number of Valuation Dates;

(e) "Pi" means the closing level of the index on the itch valuation date (i.e.: P_1 is the closing level of the index on October 6, 1999, P_2 is closing level of the index on the first Wednesday that is an Exchange Business

Day following the Trade Date and Pn is the closing level of the index on the Pinal Valuation Date.

(f) "Valuation Dates" means, commencing on October 6, 1999, and each Wednesday. thereafter up to and including the Final Valuation Date, and if any such date is not an Exchange Business Day, the next following day that is an Exchange Business Day, subject to the Market Disruption Events as set forth in the 1996 ISDA Equity Derivatives Definitions.

(g) $\sum_{i=1}^{n}$ means the summation from i=1 to i=n

Notional: 111,230,666
Equity Payment: Notional * [Final Price^2 — Initial Price^2]
If the Equity Payment is a positive value, then the Equity Payer pays the Equity Receiver this value.
If the Equity Payment is a negative value, then the Equity Receiver pays the Equity Payer the absolute value of this number

Credit Terms: n/a

In aforesaid example asset (the Standard & Poor's 5OO Composite Stock Price Index) is evaluated every week
In general case formula for Final Price calculation can be written as

$$FP^2 = \frac{\sum_{i=1}^{n-1}\left[\ln\left(\frac{P_{i+1}}{P_i}\right)\right]^2}{n-2} \times N_{year}$$

(1)

Where N_{year}=number of time interval between adjacent measurement in one year. If we get measurements every trading days, then N_{year}=252, if every week then N_{year}=52, if every month then N_{year}=12 and the like.
The pricing problem is problem of calculation of Final Price (1).
Because future values P_i are unknown stochastic variables obviously the best estimation is discounted mathematical expectation (1).
For math. expectation calculation we must know the statistical distribution of variables P_i , i=1,2…..n-1.
As usually, stocks prices and indexes are simulated by continuous diffusion process:

$$\frac{dP(t)}{P(t)} = \mu(t)dt + \sigma(t)dW(t)$$

Where
W(t)=Wiener process
$\mu(t)$=drift
$\sigma(t)$=volatility
Then by Ito's formula

$$d(\ln(P(t)) = (\mu(t) - 0.5\sigma^2)dt + \sigma(t)dW(t)$$

(2)

We suppose drift и volatility are constant (classical Black-Sholes model).

It follows from (2) variable $\ln\left(\frac{P_{i+1}}{P_i}\right)$ get normal, gauss distribution with math. expectation $(\mu - 0.5\sigma^2)(t_{i+1} - t_i)$ and variation $\sigma^2(t_{i+1} - t_i)$ and for any i this gauss variables are independent. Based on this assumptions it follows from (1)

$$E(FP^2) = \frac{\sigma^2 T + (\mu - 0.5\sigma^2)^2 \sum_{i=1}^{n-1}(t_{i+1} - t_i)^2}{n-2} \times N_{year}$$

$$T = t_n - t_1$$

(3)

On practice

$$\sigma^2 T >> (\mu - 0.5\sigma^2)^2 \sum_{i=1}^{n-1}(t_{i+1} - t_i)^2$$

$$n >> 1$$

Therefore

$$E(FP^2) \approx \frac{\sigma^2 T}{n-1} \times N_{year} = \sigma_y^2$$

(4)

where

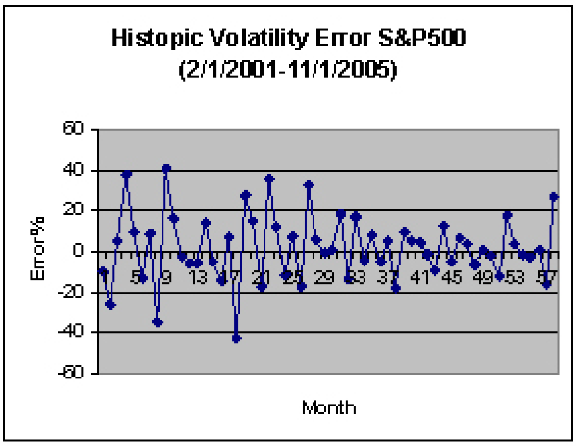
σ_y =annualized volatility.

Hereinafter we will not distinguish и σ .

Thus pricing problem reduced to building of algorithm for estimation future value of volatility . One simplest opportunity is using of historic volatility: as future volatility value we get historic volatility during the same time interval but in near past. For example, if we want to estimate value on future month today we can use previous month volatility.

Fig. 1 shows this method results for S&P500 at time interval from February 2001 until November 2005

Fig. 1



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One can see from Fig. 1 that results are not so good, but also are not so bad. The principal defect of using historic volatility is we don't use market information about future volatilities. Obviously, option's prices and implied volatilities contain this information and can be used for future volatility calculation.
We use option's prices in the following way. We consider portfolio comprising infinite large amount of Call and Put options. For strikes lesser than some quantity F we include in portfolio $\frac{dK}{K^2}$ options Put for any strike K. For strikes more than some quantity F we include in portfolio the same numberoptions Call for any strike K. This portfolio price Port equals:

$$Port = \int_0^F \frac{1}{k^2} P(k)dk + \int_F^\infty \frac{1}{k^2} C(k)dk$$

(5)

Where P (k), C (k) –premiums Put и Call, calculated by formulas:

$$P(K) = e^{-rT} \int_0^K (K - S)p(S)dS$$

$$C(K) = e^{-rT} \int_K^\infty (S - K)p(S)dS$$

(6)

In formulas (6)
r=risk-free interest rate to expiration
T=time to expiration
P(S)= statistical distribution stock prices
By substituting (6) в (5), we have

$$Port = e^{-rT}(Port1 + Port2)$$

$$Port1 = \int_0^F \frac{1}{k^2} dk \int_0^k (k - S)p(S)dS$$

$$Port2 = \int_F^\infty \frac{1}{k^2} dk \int_k^\infty (S - k)p(S)dS$$

(7)

By changing order of integration in (7) we get

$$Port1 = \int_0^F p(S)dS \int_S^F \frac{k - S}{k^2} dk$$

$$Port2 = \int_F^\infty p(S)dS \int_F^S \frac{S - k}{k^2} dk$$

(8)

It follows from (8)

$$Port = e^{-rT} \int_0^\infty p(S)dS \int_S^F \frac{k - S}{k^2} dk = e^{-rT} \int_0^\infty p(S)(Ln(F) - 1 + \frac{S}{F} - Ln(S))dS$$

(9)
Because p(S)= probability density formula (9) can be written as

$$Port = e^{-rT}(Ln(F) - 1 + \frac{E(S)}{F} - \int_0^\infty p(S)Ln(S)dS)$$

(10)

Where
E(S)=math. expectation S at expiration time
Until now we did no assumptions about statistical distribution stock prices (index). Now we suppose stock price (index) are simulated by continuous diffusion process

$$\frac{dS(t)}{S(t)} = (r - q)dt + \sigma dW(t)$$

(11)

where
W(t)=Wiener process
r=interest rate
q=dividend yield
σ=volatility
It follows from (11) that variable S is log-normal variant and its math. expectation equals

$$E(S) = S_0 e^{(r-q)T}$$

(12)

Also, it follows from (11) that integral in (10) is math. expectation of corresponding normal variable, i.e.

$$\int_0^\infty p(S)Ln(S)dS = Ln(S_0) + (r - q)T - 0.5\sigma^2 T$$

(13)

where
S0=today stock price

By substituting (12) и (13) into (9) we get

$$Port = e^{-rT}(Ln(\frac{F}{S_0}) - 1 + \frac{S_0 e^{(r-q)T}}{F} - (r - q)T + 0.5\sigma^2 T)$$

(14)

Using (14) we can calculate variance through portfolio value:

$$\sigma^2 = \frac{2}{T}(e^{rT}Port + 1 + Ln(\frac{S_0}{F}) - \frac{S_0 e^{(r-q)T}}{F} + (r - q)T)$$

(15)

Obviously, portfolio value (5) can be calculated only by using some numerical integration method. Unfortunately, on practice we have very limited amount of traded options with non zero trade volumes. For example, consider Table 1 of delayed quotes options with expiration time March 2006 (website CBOE)

Table 1

.SPX (CBOE) 1271.11 +9.02													
Dec 06 2005 @ 14:52 ET (Data 15 Minutes Delayed)													
Calls	**Last Sale**	**Net**	**Bid**	**Ask**	**Vol**	**Open Int**	**Puts**	**Last Sale**	**Net**	**Bid**	**Ask**	**Vol**	**Open Int**
06 Mar 800.0 (SPX CT-E)	0	pc	473.70	475.70	0	0	06 Mar 800.0 (SPX OT-E)	0.05	pc	0.05	0.10	0	19237
06 Mar 825.0 (SPX CE-E)	0	pc	449.10	451.10	0	0	06 Mar 825.0 (SPX OE-E)	0.50	pc	0	0.50	0	2485
06 Mar 850.0 (SPX CJ-E)	335.00	pc	424.40	426.40	0	5	06 Mar 850.0 (SPX OJ-E)	0.45	pc	0.05	0.15	0	2213
06 Mar 875.0 (SPX CO-E)	0	pc	399.70	401.70	0	0	06 Mar 875.0 (SPX OO-E)	1.20	pc	0.05	0.50	0	40
06 Mar 900.0 (SXB CT-E)	0	pc	375.10	377.10	0	0	06 Mar 900.0 (SXB OT-E)	0.25	pc	0.10	0.50	0	21421
06 Mar 925.0 (SXB CE-E)	279.00	pc	350.50	352.50	0	5	06 Mar 925.0 (SXB OE-E)	2.85	pc	0.05	0.50	0	1852
06 Mar 950.0 (SXB CJ-E)	236.00	pc	325.90	327.90	0	5	06 Mar 950.0 (SXB OJ-E)	0.60	pc	0.20	0.55	0	36005
06 Mar 975.0 (SXB CO-E)	207.50	pc	301.30	303.30	0	2	06 Mar 975.0 (SXB OO-E)	1.00	pc	0.25	0.65	0	4610
06 Mar 1000. (SPQ CT-E)	250.00	pc	276.80	278.80	0	274	06 Mar 1000. (SPQ OT-E)	0.70	-0.05	0.40	0.75	6	12019
06 Mar 1025. (SPQ CE-E)	0	pc	252.40	254.40	0	0	06 Mar 1025. (SPQ OE-E)	1.20	pc	0.60	1.10	0	16819
06 Mar 1050. (SPQ CJ-E)	187.00	pc	228.00	230.00	0	24	06 Mar 1050. (SPQ OJ-E)	1.35	-0.25	0.95	1.45	11450	37251
06 Mar 1075. (SPQ CO-E)	171.50	pc	203.80	205.80	0	1	06 Mar 1075. (SPQ OO-E)	1.90	-0.30	1.65	1.95	39	25921
06 Mar 1100. (SPT CT-E)	174.00	pc	179.90	181.90	0	906	06 Mar 1100. (SPT OT-E)	2.40	-0.50	2.00	2.80	131	49287
06 Mar 1125. (SPT CE-E)	120.00	pc	156.20	158.20	0	452	06 Mar 1125. (SPT OE-E)	3.60	-0.70	3.10	3.70	41	28823
06 Mar 1150. (SPT CJ-E)	125.00	pc	132.90	134.90	0	8915	06 Mar 1150. (SPT OJ-E)	5.00	-0.70	4.50	4.90	8018	49439
06 Mar 1175. (SPT CO-E)	102.00	pc	110.20	112.20	0	14030	06 Mar 1175. (SPT OO-E)	7.00	-1.60	6.50	6.90	65	32098
06 Mar 1190. (SPT CR-E)	0	pc	97.00	99.00	0	0	06 Mar 1190. (SPT OR-E)	12.60	pc	8.00	8.80	0	4293
06 Mar 1200. (SZP CT-E)	90.20	+4.20	88.40	90.40	74	28874	06 Mar 1200. (SZP OT-E)	10.10	-1.60	9.60	10.20	1717	55203
06 Mar 1225. (SZP CE-E)	69.50	+7.00	67.90	69.90	8	31955	06 Mar 1225. (SZP OE-E)	13.40	-3.40	13.10	14.20	903	43851
06 Mar 1250. (SZP CJ-E)	50.00	+6.00	49.20	51.20	48	36690	06 Mar 1250. (SZP OJ-E)	20.00	-3.90	19.10	20.70	1058	19906
06 Mar 1275. (SZP CO-E)	33.80	+5.30	32.80	34.80	6944	41306	06 Mar 1275. (SZP OO-E)	28.00	-5.70	27.50	29.20	1216	5486
06 Mar 1300. (SXY CT-E)	20.50	+3.90	19.90	21.50	5	63708	06 Mar 1300. (SXY OT-E)	40.30	-5.70	38.70	40.70	33	6506
06 Mar 1325. (SXY CE-E)	10.50	+1.50	10.40	12.00	5315	30325	06 Mar 1325. (SXY OE-E)	58.30	-5.50	53.90	55.90	1	201
06 Mar 1350. (SXY CJ-E)	4.80	+1.00	5.00	5.80	1940	24623	06 Mar 1350. (SXY OJ-E)	90.00	pc	72.70	74.70	0	286
06 Mar 1375. (SXY CH-E)	2.30	-0.15	2.00	2.50	55	1664	06 Mar 1375. (SXY OH-E)	134.00	pc	94.20	96.20	0	0
06 Mar 1400. (SXZ CT-E)	0.80	pc	0.60	1.10	0	40331	06 Mar 1400. (SXZ OT-E)	144.60	pc	117.50	119.50	0	352
06 Mar 1450. (SXZ CJ-E)	0.25	pc	0.10	0.35	0	8586	06 Mar 1450. (SXZ OJ-E)	171.00	-0.80	166.30	168.30	1	4953

We see a small number of options with non zero volume. But basic idea of method composition portfolio comprising Call and Put options with some weights and using this portfolio for future variance estimation can be realized for example by following way. Based on Table 1 data we include in portfolio all Call and Put options with weights proportional to trade volume. If option is not trading then volume is zero and this option is absent in portfolio.
After calculating portfolio price we calculate implied volatility. For implied volatility calculation we use Black-Sholes formula and suppose that all options have the same volatility. For Table 1 data calculated by this method future volatility equals **13.888%.**
Along with described method in which for estimation we build portfolio and calculate for this portfolio implied volatility we can consider other method in which input data are implied volatilities, calculated separately for each option with weights proportional to trade volume.
Using this data we calculate weighted average value. For Table 1 weighted average value equals **17.72453%**

Summary

We analyzed 3 potential practical estimation of future volatility or variance method.
In first method we use direct historic spot prices for calculation interesting us parameter (variance or volatilities) and make forecast future values based on historic values.
In second method we use traded options quotas, build portfolio comprised Call and Put options with some weights and for this portfolio calculate implied volatility.
Finally, in third method we use as input implied volatilities and traded volumes options and calculate weighted average value.
We also can combine all 3 methods by calculating weighted average value. Weights can be determined by back testing.

References

[1] P.Carr and R.Lee Robust replication of volatility derivatives http://www.math.nyu.edu/research/carrp/papers