

# PROBABILITY DISTRIBUTION OF COX-INGERSOLL-ROSS PROCESS

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## Abstract

The classical Cox-Ingersoll-Ross process is wide spread in theoretical finance literature. Accordingly [1] this process has the noncentral chi-square distribution. By solution of first order linear partial differential equation we calculated characteristic function of this process and compare it with known characteristic function of noncentral chi-square distribution. In general case this functions and distributions are different.

We will analyze the classical Cox-Ingersoll-Ross process for non-negative stochastic variable  $y(t)$ :

$$dy(t) = k(\eta - y)dt + \theta \sqrt{y}dW(t)$$

$$y(0) = y_0$$

(1)

Where  $W(t)$ =standard Wiener process

$k, \eta, \theta$  are constant

In financial mathematics this model was introduced by J.C.Cox, J.E. Ingersoll, Jr and S A. Ross in article "A theory of the Term Structure of Interest Rates" [1]. In this work  $y(t)$  is interest rate. The same process had

been used in Heston stochastic volatility model where  $\sqrt{y(t)}$  is volatility [2]. Probability distribution function (p.d.f) in point of time  $t$   $p(y,t)$  is solution of Kolmogorov's parabolic partial differential equation:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial y}(k(\eta - y)p) + \frac{1}{2} \frac{\partial^2}{\partial y^2}(\theta^2 yp)$$

(2)

With initial condition

$$p(0, y) = \delta(y - y_0)$$

(3)

Where  $\delta(y - y_0)$  =Dirac's function.

Let  $f(t, \omega)$  characteristic function, p.d.f. Fourier transformation

$$f(t, \omega) = \int_{-\infty}^{\infty} e^{i\omega y} p(t, y) dy$$

(4)

Function  $f(t, \omega)$  is solution of first order linear partial differential equation:

$$\frac{\partial f}{\partial t} = ik\omega f + (-k\omega + \frac{1}{2} i\theta^2 \omega^2) \frac{\partial f}{\partial \omega}$$

(5)

With initial condition

$$f(0, \omega) = e^{i\omega y_0}$$

**(6)**

Equation (5) with initial condition (6) can be solved analytically and solution is:

$$f(t, \omega) = \left(1 - \frac{1}{2}i\beta\omega\right)^{-\alpha} e^{\frac{iy_0 e^{-kt}}{\omega^{-1} - \frac{1}{2}i\beta(1-e^{-kt})}} \left(1 - \frac{\frac{1}{2}i\beta e^{-kt}}{\omega^{-1} - \frac{1}{2}i\beta(1-e^{-kt})}\right)^{\alpha}$$

$$\alpha = \frac{2k\eta}{\theta^2}$$

$$\beta = \frac{\theta^2}{k}$$

**(7)**

For given (7) characteristic function  $f(t, \omega)$  p.d.f  $p(t, y)$  can be calculated by using Fourier inversion:

$$p(t, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega y} f(t, \omega) d\omega$$

**(8)**

For given characteristic function we can easily calculate any moment of distribution using formula:

$$E(y^k) = \int_{-\infty}^{\infty} y^k p(t, y) dy = (-i)^k \left. \frac{df^k(t, \omega)}{d\omega} \right|_{\omega=0}$$

**(9)**

Differentiating logarithm function  $f(t, \omega)$  (7) we get

$$\frac{f'}{f} = \frac{0.5i\alpha\beta}{1 - 0.5i\beta\omega} + \left(1 - \frac{0.5\alpha\beta}{y_0 - 0.5\beta A}\right) \frac{dA}{d\omega}$$

$$\frac{f''}{f} - \frac{f'^2}{f^2} = -\frac{0.25\alpha\beta^2}{(1 - 0.5i\beta\omega)^2} - \frac{0.25\alpha\beta^2}{(y_0 - 0.5\beta A)^2} \left(\frac{dA}{d\omega}\right)^2 + \left(1 - \frac{0.5\alpha\beta}{y_0 - 0.5\beta A}\right) \frac{d^2 A}{d\omega^2}$$

$$A = \frac{iy_0 e^{-kt}}{\omega^{-1} - 0.5i\beta(1 - e^{-kt})} = \frac{a}{\omega^{-1} - b}$$

$$\frac{dA}{d\omega} = \frac{a}{(1 - \omega b)^2}$$

$$\frac{d^2 A}{d\omega^2} = \frac{2ab}{(1 - \omega b)^3}$$

**(10)**

From (9) and (10) we find that math. expectation and variance of  $y(t)$  equals:

$$E(y(t)) = 0.5\alpha\beta(1 - e^{-kt}) + y_0 e^{-kt}$$

$$\text{var}(y(t)) = 0.25\alpha\beta^2(1 - e^{-2kt}) + y_0\beta\left(1 - 0.5\frac{\alpha\beta}{y_0}\right)e^{-kt}(1 - e^{-kt})$$

**(11)**

Formulas (11) are identical formulas (19) [1].

Now we consider distribution law of  $y(t)$ .

Accordingly [1] “the distribution function is the noncentral chi-square,  $\chi^2[2cr(s); 2q + 2; 2u]$ , with  $2q+2$  degrees of freedom and parameter of noncentrality  $2u$  proportional to the current spot rate.”

In our notation of process formulas [1] for parameters are:

$$c = \frac{2k}{\theta^2(1 - e^{-kt})}$$

$$u = cy_0 e^{-kt}$$

$$r(s) = y$$

$$q = \frac{2k\eta}{\theta^2} - 1$$

**(9)**

Noncentral Chi-Squared distribution is distribution of sum of squares  $n$  (degree of freedom) independent normal (gauss) variables with math. expectation  $\mu$  and variance 1. Characteristic function of this noncentral chi-square equals:

$$\varphi_0(\omega, n, \lambda) = (1 - 2i\omega)^{-\frac{n}{2}} e^{\frac{i\lambda}{\omega - 2i}}$$

$$\lambda = n\mu^2$$

**(10)**

Where  $\lambda$  is parameter of noncentrality.

If variance equals  $\sigma$ , then characteristic function of this noncentral chi-square equals:

$$\varphi(\omega, n, \lambda, \sigma) = \varphi_0(\sigma^2\omega, n, \frac{\lambda}{\sigma^2}) = (1 - 2i\sigma^2\omega)^{-\frac{n}{2}} e^{\frac{i\lambda}{\sigma^2\omega - 2i\sigma^2}}$$

**(11)**

If CIR process  $y(t)$  defined by (1) has noncentral Chi-Squared distribution, then we must have opportunity choose parameters  $n$ ,  $\lambda$  and  $\sigma$  so that equations (7) and (11), functions of variable  $\omega$ , should be identical. In general case this is impossible by two reasons. First, equation (7) depends on 4 parameters and equation (11) includes only 3 parameters. Third multiplier in equation (7) is missing in equation (11). If we had neglected by third multiplier in (7) then 3 parameters in (11) should satisfied 4 equations:

$$n = 2\alpha = \frac{4k\eta}{\theta^2}$$

$$\sigma^2 = \frac{\beta}{4} = \frac{\theta^2}{4k}$$

$$\lambda = y_0 e^{-kt}$$

$$\sigma^2 = \frac{\beta}{4}(1 - e^{-kt}) = \frac{\theta^2}{4k(1 - e^{-kt})}$$

**(12)**

In general case, if  $kt \neq \infty$ , this equations are incompatible. For parameters noncentral Chi-Squared distribution determination we use 1, 3 and 4 equations (12). For given parameters we can calculate cumulative distribution function and compare it with analogous function, calculated for characteristic function (7). Cumulative distribution function for noncentral Chi-Squared distribution is well known. But we don't know analytical formulas for cumulative distribution function with characteristic function (7). This is reason why for calculation both cumulative distribution functions we will use numerical methods.

Formula (8) is equivalent to formula

$$p(t, y) = \frac{1}{\pi} \int_0^{\infty} (\cos(\omega y) \operatorname{Re}[f(t, \omega)] + \sin(\omega y) \operatorname{Im}[f(t, \omega)]) d\omega$$

(13)

Where Re[]=real part, Im[]=imaginary part.

Accumulated distribution function F(t,y) equals

$$F(t, y) = \frac{1}{\pi} \int_0^{\infty} \left( \frac{\sin(\omega y)}{\omega} \operatorname{Re}[f(t, \omega)] + \frac{1 - \cos(\omega y)}{\omega} \operatorname{Im}[f(t, \omega)] \right) d\omega$$

(14)

We will use formula (14) for calculations accumulated distribution functions for 2 characteristic functions: (7) and (11) with parameters (12). Used method of numerical integration is trapezoid rule with 3 parameters:

minimum and maximum values of  $\omega$  and number of steps n. VBA function  $\operatorname{Pr}(n, \omega_{\min}, \omega_{\max}, k, \eta, \theta, y_0, t, y)$  calculates probability of event  $y(t) < y$  at any time t for stochastic process (1) and for noncentral Chi-Squared distribution with parameters (12) (equations (1), (3) and (4)). As sample we consider stochastic process (1) with parameters from [2] (page 10)

$$y_0 = \sigma^2 = 0.06654, k = 0.6067, \eta = 0.0707, \theta = 0.2928$$

Numerical integration parameters are

$\omega_{\min} = 0.00001, \omega_{\max} = 100, n = 21$ .

Results of probability calculations for some volatility values are represented in Table 1 (t=1year)

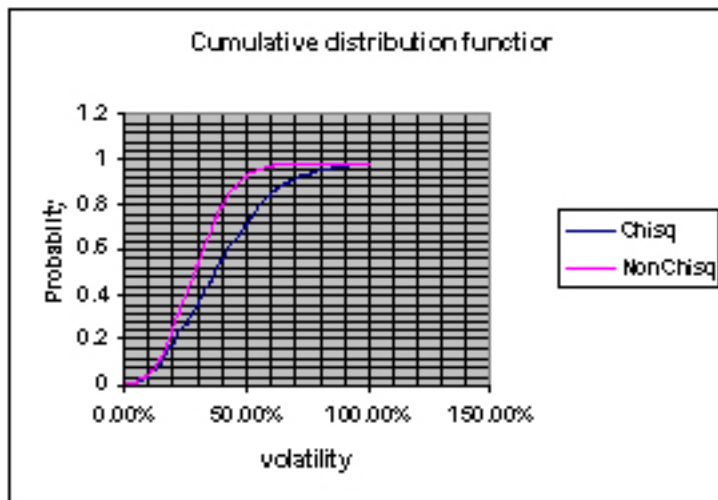
Table 1

Volatility	Pchisq	PchisqMy
1.00%	0.000256	0.000312
5.00%	0.006845	0.008442
10.00%	0.032762	0.041513
15.00%	0.089225	0.11803
20.00%	0.173994	0.243444
25.00%	0.262658	0.387523
30.00%	0.357588	0.537175
35.00%	0.456864	0.679406
40.00%	0.550824	0.791613
45.00%	0.63684	0.87154
50.00%	0.714683	0.924078
55.00%	0.780638	0.953907
60.00%	0.834487	0.969146
65.00%	0.877155	0.976346
70.00%	0.909818	0.979457
75.00%	0.933938	0.98062
80.00%	0.951254	0.981068
85.00%	0.9632	0.981194
90.00%	0.971155	0.981112
95.00%	0.976662	0.981579
100.00%	0.9795	0.980948

Where  $P_{chisq}$ =probability for noncentral Chi-Squared distribution,  $P_{chisqMy}$ = probability for stochastic process (1).

The same results are shown on Fig. 1

Fig. 1



Reference.

[1] J.C.Cox, J.E. Ingersoll,Jr and S A. Ross: "A Theory of the Term Structure of Interest Rates",Econometric a,vol.53,NO.2-March, 1985,385-407

[2] Wim Schoutens, Erwin Simons, Jurgen Tistaert: "A Perfect Calibration! Now What?,"November 20, 2003