DIFFERENCES IN THEORETICAL AND ACTUAL PRICES OF DOUBLE KNOCK-OUT AND BINARY RANGE FX OPTIONS

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On a number of occasions participants have observed a significant difference between the theoretical values of Double Knock-Out ("2KO") options and their market quotes as well as the theoretical values of Binary Range Options ("Range") and their market quotes. In general, it appears that the market quotes are significantly higher than the theoretical values and the difference becomes even more magnified when the barriers are close. The magnitude of these differences are at times as much as 5-10 times and as such are difficult to explain by factors such as market spreads and arbitrage.

Differences between theoretical values and market quotes have also been observed in in-the-money knock out options as the spot price approaches the barrier. Frequently, the explanation for this is that market-makers add a cushion of safety into the quote due to the greater risk and difficulties of delta hedging this option.

The wider discrepancies between the theoretical values and actual quotes of 2KO and Range options, in our opinion, can be partially explained by the lack of widely available pricing models for 2KO and Range options.

Due to this, many professionals have adopted approximation methods to calculate the theoretical prices for 2KO and Range options. We describe one approximation method that utilizes pricing models for knock-out options.

This approximation formula uses the following notation:

H_{low} - low barrier,

H_{up} - upper barrier,

H - barrier,

S(t) - underlying asset price at time t,

K - strike price

$2KOC(K, H_{low}, H_{up})$ - price of double knock-out call option,

$2KOP(K, H_{low}, H_{up})$ - price of double knock-out put option,

KOC(K, H) - price of knock-out call option with one barrier,

KIC(K, H) - price of knock-in call option with one barrier,

KOP(K, H) - price of knock-out put option with one barrier,

KIP(K, H) - price of knock-in put option with one barrier, C(K) - price of European call option,

P(K) - price of European put option,

Range($H_{low,}$ H_{up}) - price of Binary range option.

The approximation formula calculates the price of 2KO call and put options as:

2KOC(K, $H_{low,}$ H_{up}) = KOC(K, H_{low}) * KOC(K, H_{up})/ C(K); **(1)**

2KOP(K, $H_{low,}$ H_{up}) = KOP(K, H_{low}) * KOP(K, H_{up})/ P(K);

Since

KOC(K, H_{low}) = C(K) - KI(K, H_{low}) and

KOC(K, H_{up}) = C(K) - KI(K, H_{up})

the approximation replicates 2KO call option as:

2KOC(K, $H_{low,}$ H_{up}) = C(K) - KIC(K, H_{up}) - KIC(K, H_{low}) + KI(K, H_{up}) * KI(K, H_{low})/ C(K) **(2)**

The first three terms in the right part of equation (2) comprise portfolio(P) of long European call C(K), short knock-in call KIC(K, H_{up}) and short knock-in call KIC(K, H_{up}).

The fair value of a 2KO call represents all the positive pay-offs, at expiration, when the underlying asset stays within both barriers until expiration. Such pay-offs result from all the combination of paths that the underlying asset can take while crossing neither barrier by expiration and end up in-the-money.

To generate such paths:

take all the paths that end up in-the-money, and

subtract all the paths that cross barrier H_{up}, and

subtract all the paths that cross barrier H_{low}

Note that the paths that cross both barriers were subtracted twice and thus to generate all the paths that lead to a positive pay-off for a holder of 2KO call option:

take all the paths that end up in-the-money, and

subtract all the paths that cross barrier H_{up}, and

subtract all the paths that cross barrier H_{low}, then

add the paths that cross both barriers.

All the paths that end up in-the-money, result in a positive pay-off for the holder of a European call option. All the paths that end up in-the-money and cross barrier $H_{up,}$ result in a positive pay-off for the holder of a knock-in call option KIC(K, H_{up}). All the paths that end up in-the-money and cross barrier H_{low}, result in a positive pay-off for the holder of the of a knock-in call option KIC(K, H_{low}).

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These observations lead to equation (3):

$2KOC(K, H_{low,} H_{up}) = C(K) - KIC(K, H_{up}) - KIC(K, H_{low}) +$ *fair value of all the paths that cross both barriers*. **(3)**

From equations (2) and (3) it is logical to conclude that the accuracy of approximation (1) depends on how close the *fair value of all the paths that cross both barriers is* approximated by the term $KI(K, H_{up}) * KI(K, H_{low})/ C(K)$.

Intuitively it is clear that we should compare the underlying asset price probability of crossing both barriers $Prob(S(t) > H_{up}, S(t) < H_{low})$ with the product of two probabilities: probability of underlying asset price crossing the upper barrier $Prob(S(t) > H_{up})$ and probability of underlying asset price crossing the lower barrier $Prob(S(t) < H_{low})$.

To cross the upper barrier, the underlying asset price moves away from the lower barrier and then S(t) is *less likely* to cross lower barrier. When the distance between the lower and upper barrier is small it leads to

$$Prob(S(t) > H_{up} , S(t) < H_{low}) < Prob(S(t) > H_{up})^* Prob(S(t) < H_{low}) \qquad \textbf{(4)}$$

Formulas (2), (3) and (4) explain why when $H_{up} - H_{low}$ is small, the approximation formula (1) calculates a larger value than the theoretical model.

We use the theoretical model and approximation formulas to calculate 2KOC and 2KOP values.

To calculate the value of a Binary Range Option we use the identity that derives the price of a Range option as a function of 2KOC and 2KOP option.

$$2KOC(K= H_{low}, H_{low,} H_{up}) + 2KOP(K= H_{up}, H_{low,} H_{up})= Range(H_{low,} H_{up})^* (H_{up} - H_{low}) \qquad \textbf{(5)}$$

To prove this identity, consider the pay-off from a portfolio consisting of:

Long 2KOC with $K = H_{low,}$

Long 2KOP with $K = H_{up}$

Short Range option with $(H_{up} - H_{low})$ amount.

This portfolio will always have a zero pay-off amount at expiration.

Identity (5) can be used to realize possible arbitrage opportunities as well as to calculate the price of Binary Range Options.

For Example:

USD/DEM spot =1.5250, swap points = -80.7, DEM LIBOR = 3.50%, Expiration is in 92 days and Volatility =7.8%

For call options:

Plain vanilla call C(1.4940) = 0.036414.

Knock-Out call KOC(1.4940, 1.4940) = 0.026122

Knock-Out call KOC(1.4940, 1.557) = 0.003589

The theoretical price of 2KOC(1.4940, 1.4940, 1.557) = **0.000454.**

When the approximation method is used in formula (1)

2KOC(1.4940, 1.4940, 1.557) =0.026122*0.003589/0.036414= **0.002575**

For put options:

Plain vanilla put P(1.557) = 0.04874

Knock-Out put KOP(1.557, 1.4940) = 0.003659

Knock-Out put KOP(1.557, 1.557) = 0.036743

The theoretical price of 2KOP(1.557, 1.4940, 1.557) = **0.000476.**

When the approximation method is used in formula (1)

2KOP(1.557, 1.4940, 1.557) = (1.557, 1.4940, 1.557) = **0.002758**

Finally we use identity (5) to calculate the price of a Range option:

Theoretical value = (0.000454+0.000476)/ 0.063 = **0.014762**

Approximated value = (0.002575+0.002758)/ 0.063 = **0.08465**

Conclusion:

We have demonstrated the following:

- Described an approximation formula that is used by some market participants to price 2KO and Range options.
- Explained why and under which conditions this approximation formula will produce higher valuations than when using a model specifically designed for such purpose.
- Described several methods that may be useful to practitioners in static replication and hedging of 2KO and Range options.
- Provided an identity formula, that relates 2KOC and 2KOP to Range options. This identity formula may be used in a similar way as the Put/Call Parity.