AMERICAN EXCHANGE OPTIONS PRICING

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Abstract

We describe 3 potential methods for calculation American exchange option price. For European exchange option exists analytical formula for pricing (formula Margrabe). Unfortunately, there is no analytical formula for American option and we must use numerical method. We analyze 3 possible numerical methods. On our opinion, the best is Gauss-Hermite method of numerical integration.

Exchange Options were initially introduced by William Margrabe in his seminal 1978 paper. These type of options allows the holder of the option to exchange one asset for another and are used commonly in foreign exchange markets, bond markets and stock markets amongst others.

Let y –random variable which equals difference (spread) between two assets S(1) and S(2):

$$y = S(2) - S(1)$$

(1)

In Black-Sholes environment y equals

$$y = S_0(2)e^{x(2)} - S_0(1)e^{x(1)} \qquad \textbf{(2)}$$

where,

$S_0(i)$ -i asset's price today, (i=1,2)

x(1),x(2) are normal (Gauss)variables with math. expectation μ(i), variance $\sigma^2(i)$ and correlation coefficients ρ(1,2).

$$\mu(i) = (r_d - r_f(i) - \frac{1}{2}Vol^2(i))t$$
$$\sigma^2(i) = Vol^2(i)t \qquad \textbf{(3)}$$

where

t- time to expiration

r_d - interest rate ,

r_f - dividend rate ,

Vol - volatility .

Price euro exchange option equals

C=E[f(y)] **(4)**

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Where,
E()=math. Expectation,
f(y)=max(y,0) for call,
f(y)=max(-y,0) for put.
2-dimensional density of probability of vector (x(1),x(2)) is equal:

$$p=\frac{1}{2\pi\sigma(1)\sigma(2)\sqrt{1-\rho(1,2)^2}}e^{-\frac{1}{2(1-\rho(1,2)^2)}\left(\left(\frac{x(1)-\mu(1)}{\sigma(1)}\right)^2+\left(\frac{x(2)-\mu(2)}{\sigma(2)}\right)^2-2\rho(1,2)\frac{x(1)-\mu(1)}{\sigma(1)}\frac{x(2)-\mu(2)}{\sigma(2)}\right)},$$

(4)

Call option equals

$$C=\iint\limits_{S_0(2)e^{x(2)}-S_0(1)e^{x(1)}\geq 0}(S_0(2)e^{x(2)}-S_0(1)e^{x(1)})p(x(1),x(2),\mu(1),\mu(2),\sigma(1),\sigma(2),\rho(1,2))dx(1)dx(2)$$

(5)

From (5) we have

C=C1-C2

(6)

where

$$C1=S_0(2)e^{\mu(2)+\frac{1}{2}\sigma(2)^2}\iint\limits_{x(2)-x(1)\geq Ln(\frac{S_0(1)}{S_0(2)})}p(x(1),x(2),M2(1),M2(2),\sigma(1),\sigma(2),\rho(1,2))dx(1)dx(2)$$

$$C2=S_0(1)e^{\mu(1)+\frac{1}{2}\sigma(1)^2}\iint\limits_{x(2)-x(1)\geq Ln(\frac{S_0(1)}{S_0(2)})}p(x(1),x(2),M1(1),M1(2),\sigma(1),\sigma(2),\rho(1,2))dx(1)dx(2)$$

$$M2(1)=\mu(1)+\rho(1,2)\sigma(1)\sigma(2)$$
$$M2(2)=\mu(2)+\sigma(2)\sigma(2)$$
$$M1(1)=\mu(1)+\sigma(1)\sigma(1)$$
$$M1(2)=\mu(2)+\rho(1,2)\sigma(1)\sigma(2)$$

(7)

It is simple to show that if

$$I=\iint\limits_{y-x\geq Ln(\frac{S_x}{S_y})}p(x,y,mx,my,\sigma x,\sigma y,\rho)dxdy$$

then

$$I=Lp\left(\frac{my-mx-Ln\left(\frac{S_x}{S_y}\right)}{\sigma x^2+\sigma y^2-2\rho\sigma x\sigma y}\right)$$

where

$$Lp(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{1}{2}u^2}du$$

well known Laplace function.

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(8)

By using (3),(6),(8) we can calculate price euro exchange call option(Margrabe formula).It is obviously, that for calculation price it is necessary to know how to calculate integral:

$$Int = \iint_{S_o(2)e^{x(2)} - S_o(1)e^{x(1)} > 0} g(x(1), x(2))(p(x(1), x(2), \mu(1), \mu(2), \sigma(1), \sigma(2), \rho(1,2))dx(1)dx$$

If

$$g(x(1), x(2)) = (S_o(2)e^{x(2)} - S_o(1)e^{x(1)})$$

then (8) is analytical formula for calculation Int. For American exchange option it exists only for expiration time. Generally , function g(x,y) can be determined only numerically. Let t1 and t2- two consequent from expiration moment of time, g(u,v),f(x,y)-values of prices for t1 and t2. Then

$$f(x,y) = \iint_{\substack{u_{min} < U < u_{max} \\ v_{min} < v < v_{max}}} g(u,v)p(u, v, x + \mu(1), y + \mu(2), \sigma(1), \sigma(2), \rho(1,2))dudv$$

$$t = t1 - t2$$

$$u_{\min} = (r_d - r_f(1))T - kVol(1)\sqrt{T}$$

$$u_{\max} = (r_d - r_f(1))T + kVol(1)\sqrt{T}$$

$$v_{\min} = (r_d - r_f(2))T - kVol(2)\sqrt{T}$$

$$v_{\max} = (r_d - r_f(2))T + kVol(1)\sqrt{T}$$

(10)

where T-time from today to expiration day, k-coefficient (3-6)

We consider 3 methods of calculation integral (10).
Let

$$x(i) = x_{\min} + \frac{x_{\max} - x_{\min}}{N}(i-1)$$

$$y(j) = y_{\min} + \frac{y_{\max} - y_{\min}}{N}(j-1)$$

$$f(i,j) = f(x(i) + \frac{delx}{2}, y(j) + \frac{dely}{2})$$

$$g(k,l) = g(x(k) + \frac{delx}{2}, y(l) + \frac{dely}{2})$$

$$delx = \frac{x_{\max} - x_{\min}}{N}$$

$$dely = \frac{y_{\max} - y_{\min}}{N}$$

$$i, j, k, l = 1, 2 \ldots\ldots N+1$$

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(11)

From (10) and (11) we have

$$f(i,j)=\sum_{k}\sum_{l} g(k,l)\iint_{\substack{u(k)\le u\le u(k+1)\\ v(l)\le v\le v(l+1)}} p(u,v,x(i)+\mu(1),y(j)+\mu(2),\sigma(1),\sigma(2),\rho)\,du\,dv$$

$$\iint_{\substack{u(k)\le u\le u(k+1)\\ v(l)\le v\le v(l+1)}} p(u,v,x(i)+\mu(1),y(j)+\mu(2),\sigma(1),\sigma(2),\rho)\,du\,dv = Bnormal\left(\frac{u(k+1)-x(i)-\mu(1)}{\sigma(1)},\frac{v(l+1)-y(j)-\mu(2)}{\sigma(2)},\rho\right)-$$

$$Bnormal\left(\frac{u(k+1)-x(i)-\mu(1)}{\sigma(1)},\frac{v(l)-y(j)-\mu(2)}{\sigma(2)},\rho\right)-Bnormal\left(\frac{u(k)-x(i)-\mu(1)}{\sigma(1)},\frac{v(l+1)-y(j)-\mu(2)}{\sigma(2)},\rho\right)$$

$$+Bnormal\left(\frac{u(k)-x(i)-\mu(1)}{\sigma(1)},\frac{v(l)-y(j)-\mu(2)}{\sigma(2)},\rho\right)$$

$$Bnormal(x,y,\rho)=\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{x}\int_{-\infty}^{y} e^{-\frac{1}{2(1-\rho^2)}(u^2+v^2-2\rho uv)}\,du\,dv$$

(12)

This is first method. Actually, this is rectangular method of numerical integration.
In 2nd method we use Gaussian-Hermite numerical integration method.

$$f(x,y)=\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\sigma(1)u+x+\mu(1),\sigma(2)v+y+\mu(2))\,e^{-\frac{1}{2(1-\rho^2)}(u^2+v^2-2\rho uv)}\,du\,dv$$

Or

$$f(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\,du\int_{-\infty}^{\infty} g(\sigma(1)u+x+\mu(1),\sigma(2)(v\sqrt{1-\rho^2}+\rho u)+y+\mu(2))\,e^{-\frac{1}{2}v^2}\,dv$$

(13)

Using formulas Gauss-Hermite formulas:

$$f(x,y)=\frac{1}{\pi}\sum_{j=1}^{j=n}\sum_{i=1}^{i=n}\alpha(j)\alpha(i)\,g(\sqrt{2}\sigma(1)w(j)+x+\mu(1),\sigma(2)\sqrt{2}(\sqrt{1-\rho^2}\,w(i)+\rho w(j))+y+\mu(2))$$

where

$\alpha(i)$ – *abscissas*

$w(i)$ – *weights*

n – *order*

(14)

In this method for calculation f(i,j) if g(k,l) is given we need interpolation formula for calculation value

$$g(\sqrt{2}\sigma(1)w(j)+x+\mu(1),\sigma(2)\sqrt{2}(\sqrt{1-\rho^2}\,w(i)+\rho w(j))$$

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(15)

In 3rd method we use, that f(t,x,y) is a solution of partial differential equation:

$$\frac{\partial f}{\partial t} = \mu_1 \frac{\partial f}{\partial x} + \mu_2 \frac{\partial f}{\partial y} + \frac{1}{2} vol(1)^2 \frac{\partial^2 f}{\partial x^2} + \frac{1}{2} vol(2)^2 \frac{\partial^2 f}{\partial y^2} + \rho vol(1) vol(2) \frac{\partial^2 f}{\partial x \partial y}$$

$$\mu_2 = r_d - r_f(2) - \frac{1}{2} vol(2)^2$$

$$\mu_1 = r_d - r_f(1) - \frac{1}{2} vol(1)^2$$

(16)

Using explicit finite-difference method we received:

$$f(i,j) = pipj * g(i,j) + pip1pj * g(i+1,j) + pim1pj * g(i-1,j) + pipjp1 * g(i,j+1)$$
$$+ pipjm1 * g(i,j-1) + pro * (g(i+1,j+1) + g(i-1,j-1) - g(i+1,j-1) - g(i-1,j+1))$$

where

$$pipj = 1 - delt * \left(\frac{vol(1)^2}{delx^2} + \frac{vol(2)^2}{delx^2}\right)$$

$$pip1pj = \frac{1}{2} delt * \left(\frac{\mu_1}{delx} + \frac{vol(1)^2}{delx^2}\right)$$

$$pim1pj = \frac{1}{2} delt * \left(-\frac{\mu_1}{delx} + \frac{vol(1)^2}{delx^2}\right)$$

$$pipjp1 = \frac{1}{2} delt * \left(\frac{\mu_2}{dely} + \frac{vol(2)^2}{delx^2}\right)$$

$$pipjm1 = \frac{1}{2} delt * \left(-\frac{\mu_2}{dely} + \frac{vol(2)^2}{delx^2}\right)$$

$$pro = \frac{\rho vol(1) vol(2) delt}{4 delx dely}$$

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(17)

What method we must choose?
The criterions for selection are: error of pricing and time of calculation. Because 3hd method is the explicit finite-difference method used for initial value problem, the crucial problem is stability. For providing stability number of dividing time to expiration steps must be great .As consequence the time of calculation will be also great. Besides this, if time step is very little, then price of American option don't satisfy partial differential equation. For both 1st and 2nd methods there is no stability problem and time step can get any value. We choose time step equal 1 day.
Example.
Parameters exchange option:

$$S_0(1) = 100$$
$$S_0(2) = 100$$
$$r_d = 0$$
$$r_f(1) = 0$$
$$r_f(2) = 0$$
$$vol(1) = 10\%$$
$$vol(2) = 10\%$$
$$ro = 0$$
$$t = 10 days$$

Euro exchange option price (formula Mcgrabbe)= 0.933837253
Results of calculation and parameters are shown in Table 1.

Table 1.

Method	**Parameters**	**Price**	**Calc Time**
Rectangular	nstep=20	0.988311072	115s
Gauss-Hermite	nstep=20 ngauss=10	1.068483856	4s
Fin. differential	ntime=200 nstep=60	0.933250045	10s

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