PRICING FORMULAS FOR DOUBLE KNOCK OUT AND BINARY RANGE OPTIONS

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Payoff of Binary Range Option is

1,

conditional on Hlow < S(t) < Hup, for any t ≤T **(1)**

or 0,

conditional on S(t) Hlow or S(t)≥ Hup, for any t T **(2)**

Payoff of Double Knock Out Call option is

max(S(t)-K,0)

conditional on Hlow < S(t) < Hup, for any t T

or 0,

conditional on S(t) Hlow or S(t) Hup, for any t T

Payoff of Double Knock Out Put option is

max(K - S(t),0)

conditional on Hlow < S(t) < Hup, for any t T

or 0,

conditional on S(t) Hlow or S(t) Hup, for any t T

where:

- t - any time prior to expiration,
- T - time to expiration,
- S(T) - price of underlying at expiration,
- Hlow - lower boundary,
- Hup - upper boundary.

In the Black Sholes environment, the risk-neutral asset price process is described by the following stochastic differential equation :

dln((S(t)/S) = μ dt + σdW(t) **(3)**

where:

S(t) - asset price at time t,

S - current asset price,

$$\mu = r_d - r_f - \frac{1}{2}\sigma^2$$,

r_d - domestic interest rate,

r_f - foreign interest rate for a foreign exchange rate, carry or lease rate for commodities or divident yield for an index,

σ - volatility of asset,

W(t) - Wiener process.

Let β = ln(Hlow /S),

x_{max} =ln(Hup / Hlow)/σ,

x(t) denotes the diffusion stochastic process which ceases to exist in the points (0,) and described by the following differential equation:

dx(t)= μ/σ dt + dW(t).

Equation

ln(S(t)/S) = σ x(t) + β **(4)**

describes all the S(t) that satisfy condition (1).

Hence S(0) = S,

x(0) = ln(S/ Hlow)/ σ **(5)**

The value of a Binary Range or Double Knock Out option with time to expiration T can be written as:

V = $e^{-r_d T} E[f(y)]$, **(6)**

Where :

$f(y)$ = X{0, $x_{\max}$} for Binary Range option

y = x(T),

X{0, } = 1 if $y \in \{0, x_{\max}\}$ or

0 if $y \notin \{0, x_{\max}\}$,

$f(y) = \max(S * e^{\sigma * y + \beta} - K, 0)$ for Double Knock Out Call option

$f(y) = \max(K - S * e^{\sigma * y + \beta}, 0)$ for Double Knock Out Put option

$E[\xi]$ denotes expectation of ξ.

To solve equation (6) we utilize the fact that the function

$$\mathrm{U}(t, x) = E[f(\Re(t))]$$ **(7)**

where

$\Re(t)$ denotes the diffusion stochastic process which ceases to exist at the points 0 and 1 and is given by the following differential equation:

$d\Re(t) = g dt + dW(t)$, with the initial condition

$\Re(0) = x$ **(8)**

is described by the following partial differential equation:

$$\frac{dU(t,x)}{dt} = g \frac{dU(t,x)}{dx} + \frac{1}{2} \frac{d^2 U(t,x)}{dx^2}$$ **(9)**

with the initial condition $U(0, x) = f(x)$ and two constraints:

$$U(t, 0) = 0$$

$$U(t, 1) = 0$$

The solution for equation (9) is

$$U(T, x) = \sum_{k=1}^{\infty} \mathrm{A}_k e^{-gx - .5T(g^2 + \pi^2 k^2)} Sin(\pi k x)$$

$$\mathrm{A}_k = 2 \int_0^1 e^{gx} f(x) Sin(\pi k x) dx$$

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Given the above equations and initial condition (5) we can rewrite (6) in a closed form. Before we can do so we need to apply rescaling transformations $\Re \to x$ and some additional notation in a form of:

$\gamma = \alpha x_{\max}$

$\tau = \dfrac{T}{x_{\max}^{2}}$

$\alpha = \mu x_{\max} / \sigma$

Using the above notations the value of a Binary Range Option with time to expiration T can be written as:

$$B = \sum_{k=1}^{\infty} A_{kb} e^{-\gamma x(0) - .5\tau(\gamma^2 + \pi^2 k^2)} Sin(\pi k x(0)) \qquad \textbf{(10)}$$

The value of a Double Knock Out Call Option with time to expiration T can be written as:

$$C = \sum_{k=1}^{\infty} A_{kc} e^{-\gamma x(0) - .5\tau(\gamma^2 + \pi^2 k^2)} Sin(\pi k x(0))$$

The value of a Double Knock Out Put Option with time to expiration T can be written as:

$$P = \sum_{k=1}^{\infty} A_{kp} e^{-\gamma x(0) - .5\tau(\gamma^2 + \pi^2 k^2)} Sin(\pi k x(0))$$

Let $F(p,q,x) = \dfrac{p\sin(qx) - q\cos(qx)}{p^2 + q^2} e^{px}$

Hence $\int e^{px} \sin(qx) dx = \dfrac{p\sin(qx) - q\cos(qx)}{p^2 + q^2} e^{px} + Const$,

$$A_{kb} = 2(F(\gamma, \pi k, 1) - F(\gamma, \pi k, 0)) \qquad \textbf{(11)}$$

$$A_{kc} = 2S e^{\beta}(F(\gamma + \sigma x_{\max}, \pi k, 1) - F(\gamma + \sigma x_{\max}, \pi k, x_{str})) - 2K(F(\gamma, \pi k, 1) - F(\gamma, \pi k, x_{str}))$$

$$A_{kp} = -2S e^{\beta}(F(\gamma + \sigma x_{\max}, \pi k, x_{str}) - F(\gamma + \sigma x_{\max}, \pi k, 0)) + 2K(F(\gamma, \pi k, x_{str}) - F(\gamma, \pi k, 0))$$

where $x_{str} = \dfrac{1}{\sigma x_{\max}} Ln(\dfrac{K}{H_{low}})$

For computation purposes we need to estimate the convergence of series (10).

Let $|R(n)| = \sum_{k=n+1}^{\infty} A_k e^{-\gamma x(0) - .5\tau(\gamma^2 + \pi^2 k^2)} Sin(\pi k x(0))$

Then $|R(n)| \le 2e^{\gamma}(F - H) \sum_{k=n+1}^{\infty} e^{-.5\tau\pi^2 k^2}$

Let $D = e^{-.5\tau\pi^2}$

Hence D < 1,

$$\sum_{k=n+1}^{\infty} D^{k^2} \le D^{(n+1)^2} \frac{1}{1 - D^{2n}} \quad \text{and} \quad |R(n)| \le 2e^{\gamma}(F - H) D^{(n+1)^2} \frac{1}{1 - D^{2n}} \qquad \textbf{(12)}$$

Formulas (10), (11) and (12) comprise the close form solution for the value of a Binary Range, Double Knock Out Call and Double Knock Out Put options.

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