Digital FX option arbitrage

Mark Ioffe.
Egar Technology,Inc

We found out a theoretical possibility to build arbitrage for FX options. This possibility based on drawback Black-Sholes model. As sample, we analyze digital option, but the same is true for other FX options. For practical decision we use FOCUS system

This is a definition of digital FX option: one will receive a payout if the market is trading above or below the specified barrier at expiration. For digital FX option specification we must determine following parameters:

1. Currency 1(Cc1) and currency 2 (Cc2) which determine Foreign exchange rate (FX) or spot S. We suppose that spot (number) S has dimension $\frac{Cc1}{Cc2}$. For example, $\frac{EUR}{USD}$ or $\frac{JPY}{GBP}$.
2. Trade date and expiration date
3. Premium currency and payout currency. Payout currency is Cc1 or Cc2, Premium currency can be anyone, Cc3.
4. The specified barrier at expiration or strike K. This is number with spot's dimension.
5. Payout, the number with dimension Cc1 or Cc2. This is money that one will receive if spot is above or below strike at expiration.
6. Type of digital option: Call or Put. For Call option one will receive a payout if spot more than strike. For Put option one will receive a payout if spot lesser than strike.

How we can calculate the fair premium value? For given at expiration spot S(texp) premiums of Call C and Put P options equals:

$$C = \begin{matrix} DF * PayOut ... if .... S(t\exp) > K \\ 0.....if..S(t\exp) <= K \end{matrix}$$

$$P = \begin{matrix} DF * PayOut....if ....S(t\exp) < K \\ 0....if...S(t\exp) > K \end{matrix}$$

(1)

Where DF = discount factor for payout currency. Unfortunately, we don't know true value of S(texp). We suppose that true value is realization of stochastic variable and this stochastic variable has a continuous distribution. All we need to know for fair premium value calculation is probability density function (p.d.f) p(S). Fair premium values of Call FC and Put FP options equals:

$$FC = DF * PayOut * \text{Prob}(S(t\exp) > K) = DF * PayOut * \int_K^{\infty} p(S)dS$$

$$FP = DF * PayOut * \text{Prob}(S(t\exp) < K) = DF * PayOut * \int_{-\infty}^{K} p(S)dS$$

(2)

Accordingly Black-Sholes model logarithm of $\frac{S(t\exp)}{S}\left[\frac{Cc1}{Cc2}\right]$ has normal (gauss) distribution with math. expectation mu and mean square deviation sigma, calculated by formulas:

$$mu = (r_{Cc1} - r_{Cc2} - 0.5Vol^2) * t$$

$$sigma = Vol * \sqrt{t}$$

$$t = \frac{t\exp - t0}{365}$$

(3)

Where

r_{Cc1} =interest rate of currency 1

r_{Cc2} =interest rate of currency 2

Vol=volatility

texp=expiration date

t0=trade date

Using formulas (2) and (3) we can calculate Fair premium values of Call FC and Put FP options:

$$FC = PayOut * e^{-r_{Ccpayout} t} Lp(-u)$$

$$FP = PayOut * e^{-r_{Ccpayout} t} Lp(u)$$

$$u = \frac{Log(\frac{S}{K}) + (r_{Cc1} - r_{Cc2} - 0.5Vol^2)}{Vol\sqrt{t}}$$

$$Lp(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{1}{2}x^2} dx$$

(4)

Where $r_{Ccpayout}$ =interest rate of payout currency.

Now we consider so called reciprocal options. In our case of digital options we consider following reciprocal options. Let a pair of currencies, time to expiration and payout currency will be the same but now currency 1 will be currency 2, currency 2 will be currency 1. The dimension of Foreign exchange rate (FX) or spot P will be $\frac{Cc2}{Cc1}$. Obviously, $P = \frac{1}{S}$ and for any strike K $\Pr ob(P < K) = \Pr ob(S > \frac{1}{K})$. Consequently, if for both digital options strikes are mutually inverse then fair premium values of reciprocal options are $FC^r = FP$ and $FP^r = FC$.

307 East 53rd Street, 6th Floor
New York, NY 10022
Phone: 212-223-3552
Fax: 212-421-6608
Email: info@egartech.com
Web: www.egartech.com
Sokolniki,
4th Luchevoy prosek, pav. 5
Moscow, Russia, 107113
Phone: +7 (095) 105 33 88
Fax: +7 (095) 105 33 87
Web: www.egartech.ru

However, using for calculation of reciprocal option's fair premium values model Black-Sholes (formulas (4)) we find

$$FC^r = PayOut * e^{-r_{Ccpayout} t} Lp(-u^r)$$
$$FP^r = PayOut * e^{-r_{Ccpayout} t} Lp(u^r)$$
$$u^r = \frac{-Log(\frac{S}{K}) - (r_{Cc1} - r_{Cc2} + 0.5Vol^2)}{Vol\sqrt{t}} = -u - Vol\sqrt{t}$$
$$Lp(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{1}{2}x^2} dx$$

(5)

It follows from (4) and (5) that

$$FC^r > FP$$
$$FP^r < FC$$

(6)

The equations (6) are consequences of drawback of Black-Sholes model. Using this drawback we can build arbitrage, for example, by selling reciprocal Call and by buying Put.
As sample let our portfolio contains 2 digital options (see Table1)
Table 1

N opt	Sell/Buy	Cc1	Cc2	Payout	Cc payout	Trade Date	Exp Date	Spot	Strike	Type
1	Sell	USD	EUR	1	USD	6/14/05	9/14/05	1.2946	1.2946	Call
2	Buy	EUR	USD	1	USD	6/14/05	9/14/05	0.77224	0.7726	Put

Theoretically, this 2 option must have equal premium. Using formulas (4) and (5) we find that
Premium1=0.50
Premium2=0.48
For calculation we used following parameters:
r_{USD}=2.22%
r_{EUR}=2.13%
Vol=10.26%