APPLICATION OF FINITE DIFFERENCE METHOD FOR PRICING BARRIER OPTIONS

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Abstract

In recent years a number of authors pointed out significant stability and convergence problems while using Cox-Ross-Rubinstein binomial method to price and hedge barrier options. Different modifications were suggested to improve the convergence and stability of the binomial method. However, as this article shows, lattice approach in general has limited stability factor when applied to barrier options.

This paper suggests the use of the implicit finite difference approach in the pricing of barrier options with one or two barriers. This method has excellent stability and convergence to the solution of the underlying differential equation.

In this paper we illustrate the use of the implicit finite difference method and provide several numerical examples.

1. Introduction

Many option contract values can be obtained by solving of partial differential equations with certain initial and boundary conditions.

The finite difference approach is one of the premier mathematical tools employed to solve partial differential equations.

The finite difference approach was pioneered for valuing derivative securities by Swartz [1977] and Brennan and Swartz [1978] and has been extended by Courtadonv [1988].

There are two implementations for the finite difference method: explicit method and implicit method 1. The explicit finite difference method calculates the value of a derivative security at time t +Δt as a function of values at time t. The calculations develop recursively from time 0 to time T.

The implicit finite difference method calculates the value of a derivative security at time t as a function of values at time t + Δt. The implicit method requires solving systems of linear equations to develop calculations from time t to time t+Δt.

To compare stability of finite difference methods we classify finite difference method as unstable, conditionally stable or unconditionally stable2.

Unstable finite difference method calculates large change in option value for small change of the initial conditions (i.e. spot price). It accumulates large calculation error.
Unstable finite difference method does not converge to the solution of the partial differential equation.

Conditionally stable finite difference method calculates small change in the option value for a small change of the initial conditions. It converges to the solution of the partial differential equation, but calculation error fluctuates as a function of time and price
partitions.

Conditionally stable finite difference method calculates small change in the option value for a small change of the initial conditions. It converges to the solution of the partial differential equation and calculation error decreases when number of time and price partitions increase.

Seminal Black-Scholes [1973] equation is:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 H}{\partial S^2} + rS\frac{\partial H}{\partial S} + \frac{\partial H}{\partial T} - rH = 0, \quad \textbf{(1)}$$

where

S is the value of the underlying asset,
r is the riskless rate of interest,
T is time,
σ is the volatility of the underlying asset,
H(T,S) is the option value

Brennan and Swartz [1978] applied the

$$Y = \ln(\frac{S}{S_0})$$

transform to obtain a partial differential equation with the constant coefficients:

$$\frac{1}{2}\sigma^2 \frac{\partial^2 H}{\partial Y^2} + (r - \frac{1}{2}\sigma^2)\frac{\partial H}{\partial Y} + \frac{\partial H}{\partial T} - rH = 0 \quad \textbf{(2)}$$

where S0 is the current price of the underlying asset.

To simplify equation (2) further we follow Curtardon [1982] and define

$$X \equiv \frac{1}{\sigma}Y$$

$$V(t,X) \equiv H(T,S)e^{tr} \quad \textbf{(3)}$$

$$\gamma \equiv \frac{r}{\sigma} - \frac{1}{2}\sigma$$

where t is the time remaining to the expiration date.

Then, replacing the variables and making the necessary substitution in (2), we obtain the transformed equation:

$$\frac{\partial V}{\partial t} = \gamma\frac{\partial V}{\partial X} + \frac{1}{2}\frac{\partial^2 V}{\partial X^2} \quad \textbf{(4)}$$

Values of an option with barriers must satisfy equation (4) subject to the one or two following boundary conditions:

$$V(t, X_{\min}) = 0$$

$$V(t, X_{\max}) = 0, \quad \textbf{(5)}$$

where:

$$X_{\min} = \frac{\ln\frac{H_{low}}{S_0}}{\sigma}$$ corresponds to the low boundary,

$$X_{\max} = \frac{\ln\frac{H_{up}}{S_0}}{\sigma}$$ corresponds to the upper boundary.

The initial condition for a call option is:

$$V(0,X) = \max(S_0 e^{\sigma X} - K, 0) \quad \textbf{(6c)}$$

and for a put option is:

$$V(0,X) = \max(K - S_0 e^{\sigma X}, 0) \quad \textbf{(6p)}$$

Boyle and Lau [1994] reported that the binomial method is very unstable (“bumpy”) when used to price barrier options. They suggested the selection of a number of time partitions so that “a horizontal layer of nodes is just beyond the barrier, and is close as possible to it.” They provided examples when the modified binomial method improve convergence and stability. Ritchken [1995] notes that “...refining the partition size may not necessary produce more precise results” and suggests the use of trinomial lattice as a better solution.

Brennan and Schwartz [1978] show that the explicit finite difference method uses the same equations to relate values at two consequent time partitions as the binomial and trinomial lattice approach. Therefore binomial and trinomial lattice methods exhibit stability and convergence features of the general explicit finite difference method.

In the first section that follows we analyse applications of the finite difference method to solve equation (4) with boundary conditions (5) and initial condition (6). In the second section, we show that the explicit finite difference method is conditionally stable when applied to this task3. It explains the stability problems of the binomial method. Methods used by Boyle and Lau [1994] and Ritchken [1995] exhibit stability limitations of the explicit method as well.

Finally, we will describe how to use the implicit finite difference method to solve equation (4) with the boundary conditions (5) and initial condition (6). We use examples from Ritchken [1995] to illustrate the method and compare it with the explicit methods.

2. Implementation of the Finite Difference Method

To implement a finite difference method we construct a grid where the horizontal dimension represents discrete values of the variable X with constant step ΔX and vertical dimension represents discrete points in time with constant step Δt. In any node (i,j) we calculate time t and variable X as

t = iΔt , i = 0,...,Nt-1
X = Xmin + j ΔX, j = 0,..., NX-1

where

$$\Delta t = \frac{T_{\exp}}{N_t - 1},$$

$$\Delta X = \frac{X_{\max} - X_{\min}}{N_X - 1},$$

Xmax and Xmin correspond to boundary conditions (5), Texp is a time to expiration date, Nt is a number of time partitions, NX is a number of variable X partitions.

In such scheme, variable X lies within boundaries that correspond to conditions (5). If only one boundary exists (as it is for an option with one barrier) we define a dummy boundary as very large or very small number. In our tests we set the dummy barrier as three times greater than current asset price if the actual barrier is less than the current price or as three times less than current asset price if the actual barrier is greater than the current price 4.

To obtain a finite difference approximation to equation (4), we replace the partial derivatives in the node (i, j) by finite differences:

$$\frac{\partial V}{\partial X} = \frac{V(i,j+1) - V(i,j-1)}{2\Delta X} \quad \textbf{(7)}$$

$$\frac{\partial^2 V}{\partial X^2} = \frac{V(i,j+1) + V(i,j-1) - 2V(i,j)}{\Delta X^2} \quad \textbf{(8)}$$

Partial derivative $\frac{\partial V}{\partial t}$ can be approximated in two different ways:

$$\frac{\partial V}{\partial t} = \frac{V(i+1,j) - V(i,j)}{\Delta t} \quad \textbf{(9e)}$$

or

$$\frac{\partial V}{\partial t} = \frac{V(i,j) - V(i-1,j)}{\Delta t} \quad \textbf{(9i)}$$

Approximation (9e) is used in the explicit finite difference method and approximation (9i) is used in the implicit finite difference method.

Substituting partial derivatives in equation (4) with the approximations (7), (8) and (9e) we obtain the following explicit finite difference equation:

$$\frac{V(i+1,j) - V(i,j)}{\Delta t} = \gamma \frac{V(i,j+1) + V(i,j-1) - 2V(i,j)}{\Delta X^2} + \frac{1}{2} \frac{V(i,j+1) + V(i,j-1) - 2V(i,j)}{\Delta X^2} \quad \textbf{(10e)}$$

Substituting partial derivatives in equation (4) with the approximations (7), (8) and (9i) we obtain the following implicit finite difference equation:

$$\frac{V(i,j) - V(i-1,j)}{\Delta t} = \gamma \frac{V(i,j+1) + V(i,j-1) - 2V(i,j)}{\Delta X^2} + \frac{1}{2} \frac{V(i,j+1) + V(i,j-1) - 2V(i,j)}{\Delta X^2} \quad \textbf{(10i)}$$

Let denote

$$\alpha = \frac{\Delta t}{\Delta X^2},$$

$$p = \frac{1}{2} + \frac{1}{2}\gamma \times \Delta X$$

Then equations (10e) and (10i) can be rewritten as

$$V(i+1,j) = (1-\alpha)V(i,j) + \alpha(pV(i,j+1) + (1-p)V(i,j-1)) \quad \textbf{(11e)}$$

$$V(i-1,j) = (1+\alpha)V(i,j) - \alpha(pV(i,j+1) + (1-p)V(i,j-1)) \quad \textbf{(11i)}$$

The initial conditions (6c) and (6p) can be rewritten in terms of finite differences as:

$$V(0,j) = \max(S_0 e^{\sigma j\Delta X} - K, 0)$$
$$V(0,j) = \max(K - S_0 e^{\sigma j\Delta X}, 0) \qquad \textbf{(12)}$$

The boundary condition (5) can be rewritten in terms of finite differences as:

$$V(i,0) = 0$$
$$V(i,N_X) = 0 \qquad \textbf{(13)}$$

3. Stability of the Explicit Finite Difference Method

Approximation (11e) with conditions (12) and (13) form the explicit finite difference method. The calculations develop recursively from the expiration date (t=0) back to the current (t = Texp).

From results found in Berezin and Zidkov [1960, p.500] a necessary and sufficient condition for stability of the explicit finite difference method is :

$$\alpha \le 1 \qquad \textbf{(14)}$$

Then condition

$$\alpha = 1 \qquad \textbf{(15)}$$

sets the stability boundary for explicit finite difference method:

Explicit finite difference method is conditionally stable if $\alpha = 1$ and it is unconditionally stable if $\alpha < 1$.

Condition (14) illustrates why explicit finite difference method may produce unstable results when used to solve equation (4) with boundary condition (5).

The accuracy of the finite difference method is proportional to ΔX2. Therefore very small ΔX may be required to obtain the accurate solution. This is usually the case when the current asset price is close to the barrier. Since $\alpha = \frac{\Delta t}{\Delta X^2}$ very small, ΔX requires huge number of time partitions to satisfy condition (14). If number of time partitions is insufficiently large condition (14) may be not satisfied. This leads to an unstable solution.

Brennan and Schwartz [1978] , White and Hall [1990] show that explicit finite difference method can be equivalent to binomial lattice. To implement such an explicit finite difference method set

NX= NT,

$$\Delta t = \frac{T_{\exp}}{N_t - 1},$$

$$\Delta X = \sqrt{\Delta t}.$$

In any node (i,j) calculate time t and variable X as:

t = i Δt, i = 0, 1, ...,Nt-1

$X = (2j - N_T + i)\sqrt{\Delta t}$, j = 0, 1,...,Nt-i

To relate values at nodes i and i-1 use the following equation:

$$V(i,j) = (1-p)V(i-1,j) + pV(i-1,j-1)$$

The resulted scheme resembles binomial lattice in all the nodes.
Since $\Delta X = \sqrt{\Delta t}$ a binomial lattice satisfies condition (15) and it is conditionally stable. This explains the "bumpy" fluctuations of the results calculated by the binomial lattice method reported in Boyle and Lau [1994].

The trinomial method used in Ritchken [1995] is also explicit finite difference method with

$$\alpha = \frac{1}{\lambda^2},$$

where stretch parameter λ is chosen within [1,2] interval:

$$2 \ge \lambda \ge 1$$

This method is more stable than binomial lattice, but can require too many time partitions to obtain the stable and accurate solution. Ritchken [1995] writes that "for prices very close to the barrier 5,000 time partitions are not enough to obtain a price".

4. Implicit Finite Difference Method

Approximation (11i) with conditions (12) and (13) form the implicit finite difference method. This implicit method requires solving a group of linear equations to develop calculations from time t to time t+Δt. The calculations develop recursively from the expiration date (t=0) back to current (t = Texp). Obviously, this method is more difficult to implement than explicit methods, but it has very good stability features.
Exhibit 1 compares the convergence of the implicit method to the trinomial method. The numbers for the trinomial method are taken from Ritchken [1995]. Notice that the stretch parameter λ=1.2247 for trinomial method satisfies stability condition (14). Still implicit method converges faster and is more stable. The chart Exhibit 2 illustrates this.

From results obtained in Berezin and Zidkov [1960, p.500] a necessary and sufficient conditions for stability of implicit finite difference method is that

$$\alpha \ge 0$$

Since $\alpha = \frac{\Delta t}{\Delta X^2}$, this condition is always true. Parameters Nt and NX can be chosen independently and therefore stable and accurate solution can be produced within practical number of time partitions.

Exhibit 3 compares the prices computed by the implicit method with the prices computed by trinomial method from Ritchken[1995] when the stock price is close to the barrier.
Notice the very good convergence of the implicit method. It produces close approximation with 500 time partitions, and frequently good approximations even with 100 time partitions. When the stock price is 90.3, neither the revised binomial nor trinomial methods can produce the price with 5000 time partitions.

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5. Conclusions

We describe how to price barrier options using the finite difference methods. The major factor affecting the performance of the finite difference method is stability. We researched a stability of explicit and implicit finite difference methods when used to solve partial differential equation (4) with boundary conditions (5) and initial condition (6).

We show that any explicit method has a limited stability factor. This factor becomes crucial when very small variable step ΔX is required to obtain the accurate solution. This is the case for example when the asset price is close to the barrier. We explained the stability problems of such explicit methods as binomial lattice, revised binomial lattice and trinomial lattice.

We show how to use the implicit finite difference method to obtain stable solutions for differential equation (4) for any boundary conditions (5) and initial condition (6).

Endnotes

1. We did not illustrate use of Crank-Nicolson method in this paper. This method is second order accurate in time but more difficult to implement.
2. We are grateful to Bill Margrabe for comments on this paper, especially on stability classifications.
3. Explicit finite difference method is unconditionally stable when there is no boundary conditions (6) which is the case for vanilla options
4. Bill Margrabe points out that making the artificial boundary some number of standard deviations away from the (log of) spot price could improve results. We agree with him but we didn't try this in our tests.

Exhibit 1
Comparison of Trinomial and Implicit Down-And-Out Call Option

Case parameters
S = 95
Strike Price = 100
σ = 25% per year
Texp = 1 year
r =10% per year
Barrier=90

Number of Time Partitions	Down -and-Out Call Price (trinomial)	Down -and-Out Call Price (implicit)
50	5.9847	5.9721
75	5.9736	5.9818
100	5.9666	5.9860
150	5.9957	5.9908
200	5.9923	5.9928
400	5.9957	5.9950
800	5.9946	5.9960
1000	5.9964	5.9962
2000	5.9965	5.9965
3000	5.9967	5.9966
4000	5.9969	5.9967
Accurate value	5.9968	5.9968

The trinomial prices are taken from Ritchken [1995]
The stretch parameter λ=1.2247 for trinomial method.

Exhibit 2

Comparison of Trinomial and Implict Down-And-Out Call Option. Chart represents prices from *Exhibit 1.*

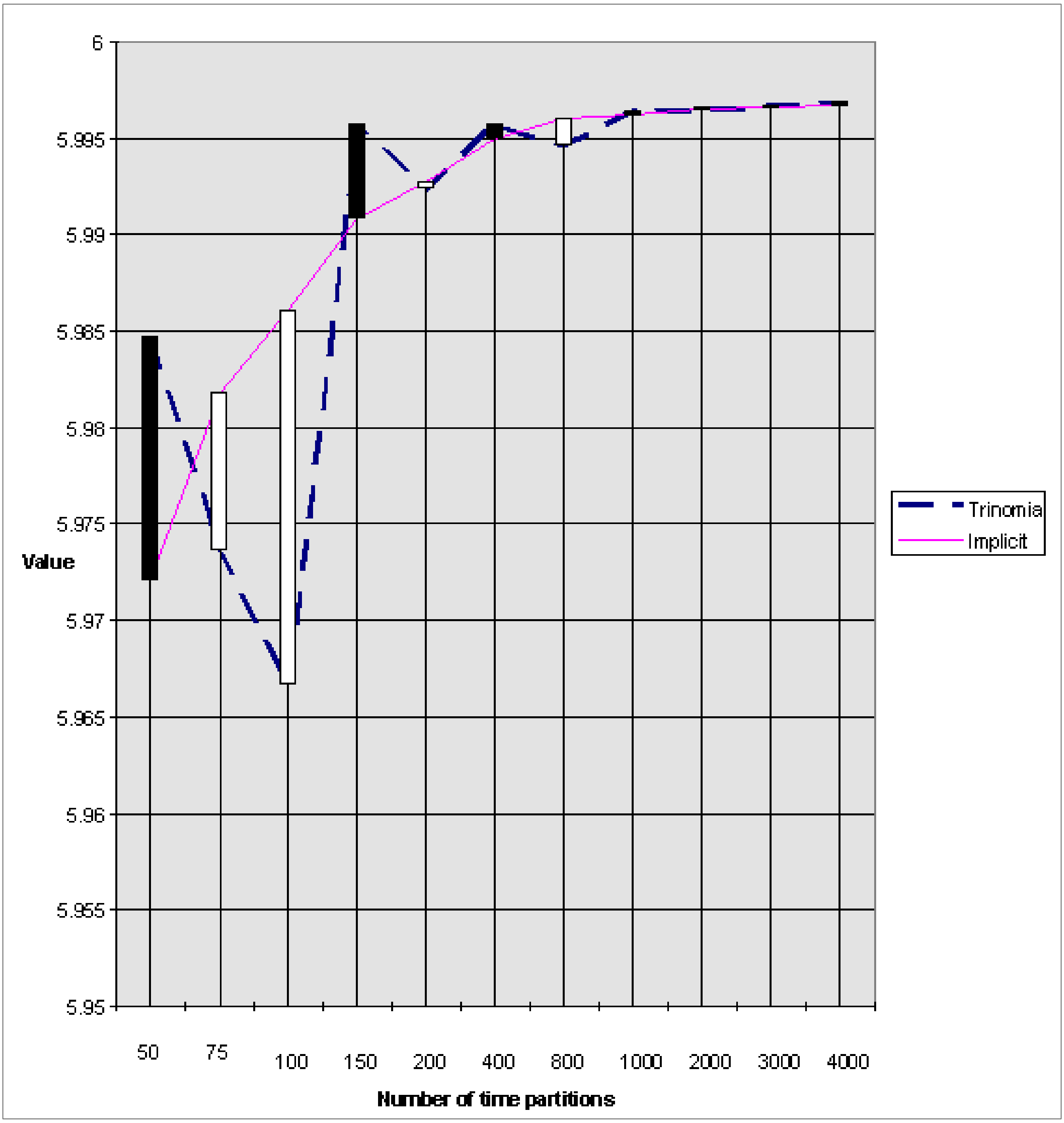

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Exhibit 3
Behavior of Barrier Option Prices Near the Boundary
The case parameters are shown in *Exhibit 1.*

Stock Price	Method	100	500	1000	2000	3000	4000	5000	True Price
94.0	Implicit	4.852	4.863	4.863	4.864	4.864			4.864
	Trinomial		4.863	4.864	4.864	4.864	4.864	4.864	
93.0	Implicit	3.690	3.701	3.701	3.701	3.701			3.701
	Trinomial		3.700	3.701	3.702	3.701	3.701	3.702	
92.0	Implicit	2.498	2.505	2.506	2.506	2.506			2.507
	Trinomial		2.504	2.506	2.506	2.506	2.506	2.506	
91.5	Implicit	1.887	1.894	1.895	1.895	1.895			1.896
	Trinomial		1.894	1.894	1.895	1.895	1.895	1.895	
91.0	Implicit	1.262	1.273	1.274	1.274	1.274			1.274
	Trinomial		-	1.274	1.274	1.275	1.275	1.274	
90.5	Implicit	0.632	0.642	0.642	0.642	0.642			0.642
	Trinomial		-	-	-	-	0.642	0.642	
90.4	Implicit	0.506	0.514	0.515	0.515	0.515			0.515
	Trinomial		-	-	-	-	0.5148	0.515	
90.3	Implicit	0.380	0.386	0.387	0.387	0.387			0.387
	Trinomial		-	-	-	-	-	-	
90.1	Implicit	0.127	0.129						0.129
	Trinomial								
90.05	Implicit	0.0634	0.0645	0.0646	0.0647				0.0647
	Trinomial								

The trinomial prices and true prices are taken from Ritchken [1995]. He does not provide trinomial method prices for 100 time partitions. He provides no prices for stock price at 90.1 and 90.05. We corrected the typo for a true price at a stock price 93.0
We didn't run the implicit method with number of time partitions greater than 3000.
The revised Binomial method from Boyle and Law [1994] requires at least 5643 time partitions when the stock price equals to 90.3

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