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BINOMIAL OPTIONS PRICING MODEL

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Abstract

Binomial option pricing model is a widespread numerical method of calculating price of American options. In terms of applied mathematics this is simple and obvious finite difference numerical method. Its basics parameters easily derived from general theory, without probabilistic considerations.

Binomial option pricing model is a widespread and in terms of applied mathematics simple and obvious numerical method of calculating the price of the American option. Price of Call options amount of money thatbuyer has to pay today for the right to buyshare at a future date at a fixed price (strike). Analogously for the Put option price is the amount of money that buyer has to pay today for the right to sell share at some future time at a fixed price. The above future date is called the expiration time. Obviously, at the expiration prices of Call and Put are:

$$C = Max(S - K, 0)$$

$$P = Max(K - S, 0)$$

(1)

Where

S - price of the share at the expiration time

K - a fixed price at which the share can be bought (Call) or sold (Put) (strike).

When buying an option of course we do not know what will be the price of the share at the expiration time. It is assumed that this price is the value of some random variable, and the price of the option is the average value of the above function, which describes the option price at the expiration timewith the discount. If we denote by p (S) density of this random variable, then the prices of European Call and Put without discount can be calculated using the following formulas:

$$C = \int_K^\infty (S - K)p(S)\mathrm{d}S$$

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$$P = \int_0^S (K - S)p(S)\mathrm{d}S$$

(2)

Thus, the only thing that should be known to calculate option prices is the probability density of future prices. Unfortunately, this is not that simple. Black and Scholes have postulated that the probability distribution of the stock price is log-normal, that is, the logarithm of the stock price has a normal distribution. This assumption is the basis of all modern option theory. Thus, in accordance with the hypothesis of the Black-Scholes, probability density of the future stock price is:

$$p(S) = \frac{1}{S\sigma\sqrt{2\pi}} e^{-(\ln S - \mu)^2/2\sigma^2}$$

(3)

Where

μ - The mathematical expectation of the logarithm of the share price;

σ -The standard deviation of the logarithm of the share price;

From the previous formulas it follows that the option prices without discount are:

$$C = e^{\sigma^2+\mu} Lp(\frac{\sigma^2 - \ln K + \mu}{\sigma}) - KLp(\frac{-\ln K + \mu}{\sigma})$$

$$P = -e^{\sigma^2+\mu} Lp(-\frac{\sigma^2 - \ln K + \mu}{\sigma}) + KLp(-\frac{-\ln K + \mu}{\sigma})$$

(4)

Where:

$$Lp(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-(u^2/2)} \mathrm{d}u$$

known as function of Laplace, the cumulative normal distribution function.

The assumption that the future price of the stock described by log-normal distribution follows from the more general assumption that the process of change in the stock price over time is a diffusion process with two constant parameters: shift μ_0 and diffusion σ_0, called in volatility in finance, i.e. equation holds:

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$$dS = \mu_0 S dt + \sigma_0 S^2 dW(t)$$

(5)

Where:

$W(t)$ - Wiener-unit variance.

$\mu_0 = r_d - r_f$

r_d - Free interest rate

r_f - Dividend interest rate

According to Ito's formula $x = \ln(S)$ satisfies the equation:

$$dx = (\mu_0 - \sigma_0^2/2)dt + \sigma_0 dW(t)$$

(6)

From this formula, it immediately follows that at time t from the purchase option $x(t) = \ln(S(t))$ has normal distribution, the mean and standard deviation is equal to:

$$\mu = \ln S_0 + (\mu_0 - \sigma_0^2/2)t$$

$$\sigma = \sigma_0 \sqrt{t}$$

(7)

Where:

S_0 - A known share price at the time of purchase of the option.

As you know, the mathematical expectation of any function from time and the trajectory of the diffusion process satisfy the differential equation in partial derivatives:

$$\frac{\partial U}{\partial t} = \mu_0 \frac{\partial U}{\partial x} + \sigma_0^2/2 \frac{\partial^2 U}{\partial x^2}$$

(8)

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It should be noted that in equation (8) time is not being counted from the moment when the deal is made, but from the expiration time. Solution of (8) with initial conditions (1) is the equations (4), in which the parameters to the equations (7).These solutions are well-known Black-Scholes formulas calculate the price of European Call and Put when multiplied by $e^{-r_d t}$. For the numerical solution of equation (8) we can use the correspondingfinite difference method. In the simplest case, the first and second partial derivatives are approximated by the following finite differences:

$$\frac{\partial U(t,x)}{\partial t} = \frac{U(t+\delta t,x) - U(t,x)}{\delta t}$$

$$\frac{\partial U(t,x)}{\partial x} = \frac{U(t,x+\delta x) - U(t,x-\delta x)}{2\delta x}$$

$$\frac{\partial^2 U(t,x)}{\partial x^2} = \frac{U(t,x+\delta x) + U(t,x-\delta x) - 2U(t,x)}{\delta x^2}$$

(9)

Substituting (9) into (8), we obtain the following calculation formula to pass the time t to $t + \delta t$:

$$U(t+\delta t,x) = U(t,x)(1-\sigma_0^2\frac{\delta t}{\delta x^2}) + 0.5U(t,x+\delta x)(\mu_0\frac{\delta t}{\delta x} + \sigma_0^2\frac{\delta t}{\delta x^2}) + 0.5U(t,x-\delta x)(-\mu_0\frac{\delta t}{\delta x} + \sigma_0^2\frac{\delta t}{\delta x^2})$$

(10)

Note that formula (10) refers to the so-called explicit finite difference schemes, [1] in which the values of a subsequent layer calculated directly by the values of the previous layer. In the so-called implicit schemes for finding the values of the function at the next layer has to solve a system of linear equations. The advantage of the explicit scheme is reduced compared with the implicit number of calculations. The disadvantage is that such a scheme may be unstable, as occurs, for example, by using the binomial method for options with barriers. For implementation (10) necessary to choose two parameters: time step δt and space δxstep . In the binomial method is chosen only time step. Just select the number of steps from 0 to expiration, and the time step is:

$$\delta t = \frac{t}{n}$$

(11)

Space step is chosen so that the transition from the previous step to the subsequent used not three values offunction but two. The method itself is called binomial because of this circumstance. To do this, it is assumed that:

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$$1-\sigma_0^2\frac{\delta t}{\delta x^2}=0$$

From this it follows that the space step is:

$$\delta x=\sigma_0\sqrt{\delta t}$$

(12)

With (12) calculation scheme (10), implemented in binomial is:

$$U(t+\delta t,x)=0.5U(t,x+\delta x)(\frac{\mu_0\sqrt{\delta t}}{\sigma_0}+1)+0.5U(t,x-\delta x)(-\frac{\mu_0\sqrt{\delta t}}{\sigma_0}+1)$$

(13)

Formula (13) is obtained without any probabilistic considerations on the basis of the well known methods for the numerical solution of the differential equation in partial derivatives. However, it can easily be interpreted in terms of the theory of probability. Indeed, from (13) it follows that the price of an option at a subsequent point in time is the mathematical expectation of option prices in the two neighboring grid points, down one step and up one step. The transition probabilities of the nodes up and down are appropriate coefficients in (13). That is

$$pup=0.5(\frac{\mu_0\sqrt{\delta t}}{\sigma_0}+1)$$

$$pdown=0.5(-\frac{\mu_0\sqrt{\delta t}}{\sigma_0}+1)$$

(14)

If we define the spatial nodes not on log stock prices, but on the very prices, the upper and lower values of the prices of shares related to the value where there is a movement depends on:

$$Sup=Se^{\delta x}$$

$$Sdown=Se^{-\delta x}$$

(15)

It is interesting to note how artificially (14) and (15) are derived, forming the basis of the binomial method, in the so-called financial mathematics, and, in particular, by the authors of the method [2]. Note also that according to (13) in the binomial method one uses not all the nodes in time and space of rectangular grid, but the so-called triangular tree, at the foundation of which lies the point in time for which the option price is calculated and at each time step function values are considered to be only half of the spatial units. Obviously, finite difference method can be used for the European option. But in this case there is an explicit analytical formula (Black-Scholes) to make it impractical. In the case of American option, value from the option pricing formula (13) is compared to the value obtained in early exercise that is the difference between the stock price and the strike price for Call and difference between strikes and the stock price for Put. If these differences exceed values from the option pricing formula (13), the latter is replaced by the corresponding difference.

References.

[1] G. Ioffe, M. Ioffe APPLICATION OF FINITE DIFFERENCE METHOD FOR PRICING BARRIER OPTIONS. http://www.egartech.com/docs/finite_difference_barrier_options.pdf

[2] John C. Cox, Stephen A. Ross, and Mark Rubinstein. 1979. "Option Pricing: A Simplified Approach." *Journal of Financial Economics*7: 229-263.