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Calculation of the integral required to calculate Asian option

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Abstract

The article refers to the calculation of the cost of Asian option. In particular, using the proposed M. Curran account for the fact that the Arithmetic Mean is always no less than Geometric Mean. When calculating the integral occurs associated with a multidimensional normal distribution. The article presents a method for calculating the integral that is different from the one proposed by M. Curran and is much simpler.

Payoff at expiration T for Asian Call option equals:

$$C = Max(\frac{1}{n}\sum_{i=1}^{n} S(t_i) - K, 0)$$

(1)

Where

$S(t_i)$ – stock price at time t_i;

K – Strike price;

n – number of averaging.

In accordance with the Black-Sholes model price change stock S (t) is given by:

$$dS(t) = S(t)(r_d - r_f)dt + S(t)\sigma dW(t)$$
$$0 \le t \le T$$

(2)

Where

r_d = constant interest rate;

r_f = constant dividend rate;

σ = constant volatility.

That is payoff at expiration T equals:

$$\Pr C = Max(\frac{1}{n}\sum_{i=1}^{n} e^{x(i)} - K, 0)$$

(3)

Where x (i) = Ln (S (i)), i = 1,2 ... n the values of the logarithms of the respective stock prices at averaging time t (i). In accordance with the Black-Sholes model the random variables x (1), x (2) ... .

x (n) form a normal, Gaussian vector with the vector of the expectation of m (i) and covariance matrix Covmat (i, j), calculated according to formulas:

$$m(i) = (r_d - r_f - 0.5\sigma^2)t(i)$$
$$Covmat(i,j) = \sigma^2 Min(t(i), t(j))$$

(4)

Non arbitrage option price is the expectation of the random variable PrC from the formula (3).

$$\Pr Call = e^{-r_d T}\int_{-\infty}^{\infty}....\int_{-\infty}^{\infty} Max(\frac{1}{n}\sum_{i=1}^{n} e^{x(i)} - K, 0)\, p(x(1), x(2)...x(n), m, Covmat)dx(1)dx(2)...dx(n)$$
$$p(x(1), x(2)...x(n), m, Covmat) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{Det(Covmat)}} e^{-\frac{1}{2}(x-m)^T Covmat^{-1}(x-m)}$$

(5)

Method of calculating non arbitrage price of the option proposed by Michael Curran, based on a simple fact: the Arithmetic Mean of all n values is always no less than the Geometric Mean of these same quantities. This fact is the following way can be used to calculate a price.

$$\Pr Call = \Pr Call1 + \Pr Call2$$
$$\Pr Call1 = e^{-r_d T}\int_{\frac{1}{n}\sum_{i=1}^{n} x(i) > LnK}.....\int (\frac{1}{n}\sum_{i=1}^{n} e^{x(i)} - K)\, p(x(1), x(2)...x(n), m, Covmat)dx(1)dx(2)...dx(n)$$
$$\Pr Call2 = e^{-r_d T}\int_{\frac{1}{n}\sum_{i=1}^{n} x(i) < LnK}.....\int Max(\frac{1}{n}\sum_{i=1}^{n} e^{x(i)} - K, 0)\, p(x(1), x(2)...x(n), m, Covmat)dx(1)dx(2)...dx(n)$$
$$G = \left(\prod_{i=1}^{n} e^{x(i)}\right)^{\frac{1}{n}}$$

(6)

Integral PrCall1 can be written as:

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$$\Pr Call1 = e^{-r_d T} \frac{1}{n}\sum_{i=1}^{n} \int\limits_{\frac{1}{n}\sum_{i=1}^{n} x(i) > LnK} ....\int (e^{x(i)} - K)\, p(x(1), x(2)...x(n), mCovmat)dx(1)dx(2)...dx(n)$$

(7)

Thus, there is the problem of computing the integral:

$$Int = \int\limits_{\frac{1}{n}\sum_{i=1}^{n} x(i) > LnK} ....\int e^{x(k)} p(x(1), p(x2)....x(n), m, Covmat)dx(1)dx(2)....dx(n)$$

(8)

It can be shown that:

$$e^{x(k)} p(\overline{x}, m, Covmat) = C(k)\, p(\overline{x}, M, Covmat)$$

$$M = m + Covmat\begin{pmatrix} i(1) = 0 \\ i(2) = 0 \\ ... \\ i(k) = 1 \\ ... \\ i(n) = 0 \end{pmatrix} = m + \begin{pmatrix} Covmat(1,k) \\ Covmat(2,k) \\ ... \\ Covmat(k,k) \\ ... \\ Covmat(n,k) \end{pmatrix}$$

$$C(k) = e^{\frac{1}{2}(m-M)^T Covmat^{-1}(m-M) \; + m(k)} = e^{\frac{1}{2}Covmat(k,k) + m(k)}$$

(9)

With (9), the formula (8) can be written as:

$$Int = C(k) \int\limits_{\frac{1}{n}\sum_{i=1}^{n} x(i) > LnK} ....\int \; p(x(1), p(x2)....x(n), M, Covmat)dx(1)dx(2)....dx(n)$$

(10)

Let

$$y = \frac{1}{n}\sum_{i=1}^{n} x(i)$$

(11)

Math expectation Ey and dispersion σ_y^2 y equals:

$$Ey = \frac{1}{n}\sum_{i=1}^{n} M(i)$$

$$\sigma_y^2 = \sum_{i=1}^{n} Covmat(i,i) + 2\sum_{i=1}^{n}\sum_{j>i}^{n} Covmat(i,j)$$

(12)

Int in formula (10) obviously equals:

$$Int = C(k)\Pr obability(y > LnK) = C(k)Lp(-\frac{LnK - Ey}{\sigma_y})$$

$$Lp(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}u^2} du$$

(13)

In [1] M. Curran offers another way to calculate by conditioning on the Geometric Mean.

Let (S_x,S_y)- two-dimensional log-normal variable, that is, the density distribution of this variable is described by the formula:

$$p(S_x, S_y) = \frac{1}{S_x S_y} dnorm2(LnS_x, LnS_y, mx, my, \sigma x, \sigma y, \rho)$$

(14)

Where $dnorm2(LnS_x, LnS_y, mx, my, \sigma x, \sigma y, \rho)$ the density of two-dimensional normal distribution:

$$dnorm2(LnS_x, LnS_y, mx, my, \sigma x, \sigma y, \rho) = \frac{1}{2\pi\sigma x\sigma y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\left(\frac{LnS_x - mx}{\sigma x}\right)^2 + \left(\frac{LnSy - my}{\sigma y}\right)^2 - 2\rho\frac{LnS_x - mx}{\sigma x}\frac{LnS_y - my}{\sigma y}\right)} \quad (15)$$

The density distribution of S_x is described by the formula:

$$p(S_x) = \frac{1}{S_x} dnorm1(LnS_x, mx, \sigma x)$$

(16)

Where $dnorm1(LnS_x, mx, \sigma x)$ the density of normal distribution:

$$dnorm1(LnS_x, mx, \sigma x) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{1}{2}\left(\frac{LnS_x - mx}{\sigma x}\right)^2}$$

(17)

The conditional distribution S_y / S_x has density distribution:

$$p(S_y / S_x) = \frac{p(S_x, S_y)}{p(S_x)} = \frac{1}{S_y} \frac{dnorm2(LnS_x, LnS_y, mx, my, \sigma x, \sigma y, \rho)}{dnorm1(LnS_x, mx, \sigma x)}$$

(18)

In accordance with the well-known property of a bivariate normal distribution, we have:

$$\frac{dnorm2(LnS_x, LnS_y, mx, my, \sigma x, \sigma y, \rho)}{dnorm1(LnS_x, mx, \sigma x)} = dnorm1(LnS_y, my + \rho\frac{\sigma y}{\sigma x}(LnS_x - mx), \sigma y\sqrt{1-\rho^2}))$$

(19)

From formulas (18) and (19) we deduce that the conditional distribution S_y / S_x is a log-normal and

$$E(S_y / S_x) = e^{my + \rho\frac{\sigma y}{\sigma x}(LnS_x - mx), \sigma y + \frac{1}{2}(1-\rho^2)\sigma_y^2}$$

(20)

Let:

$$S_x = G = e^{\frac{1}{n}\sum_{i=1}^{n} x(i)}$$

$$S_y = S(k) = e^{x(k)}$$

(21)

In this case using formula (12):

$$mx = \frac{1}{n}\sum_{1}^{n} m(i)$$

$$\sigma x^2 = \sigma_y^2$$

$$my = m(k)$$

$$\sigma y = \sqrt{Covmat(k,k)}$$

$$\rho = \frac{1}{n\sigma x \sigma y}\sum_{i=1}^{n} Covmat(i,k)$$

(22)

Then:

$$Int_1 = \int_K^\infty E(S(k)/G)p(G)dG = \int_K^\infty e^{my+\rho\frac{\sigma y}{\sigma x}(LnG-mx),\sigma y+\frac{1}{2}(1-\rho^2)\sigma_y^2} \frac{1}{\sqrt{2\pi}\sigma x G} e^{-\frac{1}{2}\left(\frac{LnG-mx}{\sigma x}\right)^2} dG$$

(23)

Let

$$u = \frac{LnG - mx}{\sigma x}$$

(24)

Then from (23):

$$Int1 = e^{my+\frac{1}{2}(1-\rho^2)\sigma_y^2} \int_{\frac{LnK-mx}{\sigma x}}^{\infty} e^{\rho\sigma y u} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du = e^{my+\frac{1}{2}\sigma_y^2} Lp\left(-\frac{LnK - mx - \rho\sigma y\sigma x}{\sigma x}\right)$$

(25)

Given the formulas (9), (12) and (22) formula (25) can be written as:

$$Int1 = e^{my+\frac{1}{2}\sigma_y^2} Lp\left(-\frac{LnK - Ey}{\sigma x}\right)$$

(26)

Thus both methods compute the integral coincide. Obviously, the first method is much simpler than the second.

Reference.

[1] Michael Curran. Valuing Asian and Portfolio Options by Conditioning on the Geometric Mean Price. Portfolio Engineering. NOV-18-1993